THE SECULAR EFFECTS OF THE INCREASE OF THE SUN'S MASS UPON THE MEAN MOTIONS, MAJOR AXES AND ECCENTRICITIES OF THE ORBITS OF THE PLANETS.

By T. J. J. SEE.

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In the days of Newton, Lagrange and Laplace, it was assumed that the formation of the planetary system was essentially complete, and the sun's attraction rigorously constant from age to age; and it was scarcely deemed necessary to consider the secular effects of slight modifying causes such as the downfall of cosmical dust upon the bodies composing the solar system. But the progress of the past century has shown that the Newtonian hypothesis of a constant mass and a central attraction depending wholly on the distance, but not on the time, is at best a very rough approximation to the truth; for in addition to the downfall of cosmical dust upon all the bodies of our system, it has been shown by the researches of Arrhenius, Schwartzchild and others, that the sun especially is losing finely divided matter under the action of repulsive forces such as we see illustrated in the streamers of the corona and the tails of comets. In our modern studies of the orbital motions of the heavenly bodies, therefore, we have to take the central mass as variable with the time, and consider the small secular changes which will follow from a variation of the central attraction incident to a gradual change of mass.

These questions have been treated in some form by many of the successors of Newton; and even this great philosopher himself in one case supposed that the central mass might be varied by a comet falling into the sun.[1] Laplace devotes considerable attention to the secular equations for determining the effects of the decrease of the sun's mass due to loss of light, then supposed to be of corpuscular

[1] "Principia," Lib. III., last proposition.

character.[2] The modern discussions based on the analytical methods of Gyldén are, however, much more satisfactory than those of the age of Laplace; and I propose to give a brief account of them, chiefly with a view of summarizing the state of our knowledge, and of removing some inconsistencies which may mislead those who are unfamiliar with the literature of the subject.

For example, in the late Professor Benjamin Peirce's "Ideality in the Physical Sciences," Boston, 1881, p. 131, the following curious statement occurs:

> The constant increase of the solar mass would have an influence on the planetary orbits. It would diminish their eccentricities, according to a law of easy computation. Hence it is possible that the orbits of the planets may have been originally very eccentric, almost like those of the comets; and their present freedom from eccentricity may have resulted from the growing mass of the sun. What modification of the nebular theory may be involved in this supposition cannot easily be imagined, without the guidance of some indication from nature.

This statement is misleading and erroneous, and the only way I can explain its appearance in the writings of Peirce is by the fact that his last lectures were prepared when he was at an advanced age and in ill health; and thus it is probable that some confusion occurred. Quite recently an analogous confusion has appeared in the *Astronomische Nachrichten,* No. 4454, in a short article by Dr. R. Bryant, on the secular acceleration of the moon's mean motion.

In order to place before the reader a summary of the chief investigations bearing on the problems now under discussion we cite the following papers:

1. "The Problem of the Newtonian Attraction of two Bodies with masses Varying with the Time," H. Gyldén (*A. N.,* 2593).

2. "Ein Specialfall des Gyldén'schen Problems," J. Mestschersky (*A. N.,* 3153 and 3807).

3. "Ueber Central Bewegungen," R. Lehmann-Filhés (*A. N.,* 3479–80).

4. "Note on Gyldén's Equations of the Problem of Two Bodies with Masses Varying with the Time," E. O. Lovett (*A. N.,* 3790).

5. "Ueber die Bedeutung Kleiner Massenänderungen für die Newtonsche Central Bewegung," Dr. E. Strömgren (*A. N.,* 3897).

[2] Mécanique Céleste, Liv. X., § 20.

The last of these papers is the most important, since it supplements and extends the results of the earlier investigators. Professor Strömgren's method is one of great generality and appears to be the most satisfactory yet devised; and we shall base our brief discussion chiefly on this paper.

If σ be a very small quantity, and $\phi(t)$ some function of the time, the original unit of mass becomes $1 + \sigma\phi(t)$, and the differential equations of motion become

$$\left.\begin{aligned} \frac{d^2x}{dt^2} + k^2[1 + \sigma\phi(t)]\frac{x}{r^3} = 0, \\ \frac{d^2y}{dt^2} + k^2[1 + \sigma\phi(t)]\frac{y}{r^3} = 0; \end{aligned}\right\} \tag{1}$$

where k^2 is the gravitation constant, and the mass is unity at the initial epoch $t = 0$.

The new constant of areas becomes

$$x\frac{dy}{dt} - y\frac{dx}{dt} = k\sqrt{[1 + \sigma\phi(t)]p} = \text{const.} \tag{2}$$

Other formulæ of interest are:

$$V^2 = k^2[1 + \sigma\phi(t)]\left(\frac{2}{r} - \frac{1}{a}\right), \tag{3}$$

$$\delta a = a_0\sigma\phi(t) - 2a^2_0\sigma\int_0^t \phi'(t)\frac{1}{r}dt, \tag{4}$$

$$\frac{1}{r} = \frac{1}{a}[1 + 2\Sigma J_i(ie)\cos i(e + nt - \pi)] \tag{5}$$

And finally after a careful investigation of all effects due to errors of the first order of the disturbing force, $\sigma\phi(t)$, Strömgren finds:

$$\left.\begin{aligned} \delta a &= -a\sigma\left[t + 2\frac{e}{n}(\sin E - \sin E_0)\right], \\ \delta e &= -\frac{1 - e^2}{n}\sigma(\sin E - \sin E_0), \\ \delta\pi &= \frac{\sqrt{1 - e^2}}{en}\sigma(\cos E - \cos E_0). \end{aligned}\right\} \tag{6}$$

Here n is the mean motion and E the eccentric anomaly. It will be seen from the first of equations (6) that the semi-axis major is diminished by a secular term depending on t, and by a periodic term depending on the difference of the sines of the angles E and E_0, or the position in the orbit. Thus the mean distance is subjected to both periodic and secular variation.

In the case of the eccentricity, however, the second of the equations (6) shows that there is no secular term, and only periodic changes occur. A similar remark applies to the longitude of the perihelion as shown by the third equation of (6).

We conclude, therefore, from Strömgren's careful analysis that there is no secular decrease in the eccentricity due to a steady growth of the central mass; and that the views expressed by Peirce and Bryant are due to confusion, or to some error in the chain of reasoning.

This conclusion accords with the result reached by Professor Lehmann-Filhés, in paper No. 3,[3] cited above. For Lehmann-Filhés shows that

$$\left.\begin{aligned} e \cos \pi &= e_0 \cos \pi_0 + \text{periodic terms}, \\ e \sin \pi &= e_0 \sin \pi_0 + \text{periodic terms}; \end{aligned}\right\} \qquad (7)$$

and remarks that when the attracting mass slowly increases the orbit slowly narrows up, but yet always remains a similar conic section. He adds that this is true for any eccentricity whatever. The results of Lehmann-Filhés and Strömgren, each worked out independently of the other, and with much detail, are therefore in entire accord; and as Strömgren's development is given in full, and every step in his analysis is quite clear, we must reject the conclusions of Peirce and Bryant as not well founded.

This concllusion that the *steady increase* of the central mass will not diminish the eccentricity also confirms the results reached by Airy[4] and by Sir John Herschel.[5] For these eminent authorities show that a central attractive disturbance decreases the eccentricity as the planet moves from the perihelion to the aphelion, but increases

[3] Cf. *A. N.*, 3479–3480.

[4] "Gravitation," pp. 50–51.

[5] "Outlines of Astronomy," tenth edition, 1869, p. 463.

it correspondingly in going from the aphelion to the perihelion; so that only periodic changes of the elements e and π occur.

Accordingly it follows that the only possible cause which could have diminished and practically obliterated the eccentricities of the orbits of the planets and satellites is the secular action of a resisting medium, as fully set forth in Volume II. of my "Researches on the Evolution of the Stellar Systems," 1910. Increasing the central mass accelerates the mean motions, and thus becomes very sensible in the theory of the motions of the planets; but it has no effect on the shape of their orbits. The almost circular form of the planetary orbits, therefore, may be referred to the secular action of a resisting medium and to no other cause whatsoever.

This result is of no ordinary interest, since it refers the roundness of the planetary orbits to but a *single physical cause,* and gives us what mathematicians call a *unique solution* of the leading problem of cosmogony. For Babinet's criterion shows beyond doubt that the planets never were detached from the central bodies which now govern their motions; and the argument given in Volume II. of my "Researches" proves that all these bodies were formed in the distance and afterwards neared the central masses about which they now revolve. The demonstration of the true mode of formation of our solar system is therefore supported by the necessary and sufficient conditions usually required in mathematical reasoning; and we may say that the laws of the formation of the solar system have been confirmed by mathematical criteria having all the rigor required in the science of geometry. This generalization will, I think, add not a little to our interest in the geometry of the heavens; and it is equally worthy of the attention of the astronomer, the geometer and the natural philosopher, who so long struggled to unfold the wonderful process involved in the formation of the planetary system.

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